About the Mathematics: Multi-Digit Addition & Subtraction Strategies

An overview of the multi-digit addition & subtraction strategies that emerge from student work

Bridging From the Model to Number Strategy

The focus here is how students bridge from their use of physical models to the number level. Advanced students benefit from seeing the connection between these two modes of representation as much as emergent students. Bridging between these two modes builds the necessary conceptual understanding students need to fluently work with multi-digit numbers. Students need to move beyond direct modeling and become knowledgeable in the use of multiple abstract strategies. First through third grade is the point in students learning that the abstract strategies need to become solidified for multidigit addition and subtraction. Less time is spent, if at all, in any succeeding grades on these two operations. Grounding students in truly understanding why these strategies work and the different mathematical ideas upon which they are based is essential.

Decomposition and Reconfiguring of Number

If you don't like the numbers you are given, break them apart or change them to make them easier to work with! This statement to students captures the essence of the decision making process they need to go through when operating on multi-digit numbers. In addition with whole numbers, the decomposition options focus on combining larger combinations. Organizing around landmarks of ten, place value components, and incrementing decades from any number are typical options behind decomposition decisions. For example, solve 54 + 38. I can break the 54 into 50 + 4 and 38 into 30 + 8 to combine like units. I can keep the 54 whole and only decompose 38 into 30 + 6 + 2 thus solving the problem as 54 + 30 ⟶ 84 + 6 ⟶ 90 + 2 ⟶ 92. The decomposition in that sequence was based on place value [38 = 30 + 8], reconfiguring a number through incrementing by decades from any number [54 + 30 = 84], and organizing around a landmark of ten [8 = 6 + 2 in order to solve 84 + 6 = 90].

In subtraction, similar decomposition and reconfiguring of number occurs as well. The question is which number. In the example of 54 – 38, are you going to break the 54 apart into 40 + 14 as in the American Standard Algorithm or the 38 into 30 + 4 + 4 as in the Counting Back or Incremental strategy? In either case, one of the numbers needs to be broken apart. If you decide to *change the numbers to make them easier to work with*, you need to make sure to *keep it equal.* 54 – 40 is easier to work with but it is not equal to 54 – 38. However, 38 does equal 40 – 2. A little intuitive understanding of the order of operations is the reason why 54 – 38 = 54 – (40 – 2) = 54 – 40 + 2. More realistically from a student's point of view, by adding 2 to 38 you took 2 too many away, so you need to add back that 2 to keep it equal. If understanding subtraction as the difference between two numbers is understood, then keeping the equation equal by adding 2 to both numbers also makes the numbers easier to subtract. Thus 54 – 38 = (54 + 2) – (38 + 2) = 56 – 40. In all of the examples here, understanding which numbers to break apart or adjusting the numbers are key mathematical concepts to comprehend in multidigit addition and subtraction. Equality is also imperative because if you are breaking numbers apart or changing them, the resulting changes had better be equivalent mathematical expressions.

Developmental Range of Solution Strategies

Out of students' direct modeling approaches arise the abstract number strategies. Early number strategies reflect skip counting one ten at a time. These are all natural progressions in student development. These should be allowed. Particularly in the fall of the year, students need to progress through the more rudimentary versions of strategies en route to more fluent usage of abstract strategies. It is over the arc of the school year and the vertical progression across grade levels that students mature in their strategy use. As teachers, our job is to attend to students' thinking, interpet what the next steps in their development are, and then intentionally plan the warm-ups and problem-solving tasks that help refine the number sense and fluency levels that are needed.

Flexible Use of Strategies – The Idea of Efficiency

Efficiency is not the same as speed. A person can be very fast counting by ones on their fingers, but that is not considered an efficient strategy. Efficiency is about making judgments about which strategy to use given the numbers in the context in which you are working. I could use the American Standard Algorithm to solve 400 – 296 but most adults when given this number without pencil and paper typically either Add Up To or Count Back To. Example: 296 + 100 ⟶ 396 + 4 ⟶ 400, the answer is 100 + 4 = 106. That is less work than breaking 400 into 300 + 100 then 300 + 90 + 10 to then do 10 – 6, 90 – 90, 300 – 200 to get an answer of 106. That strategy works, but it is more work than adding up given the numbers in this particular problem. 106 – 29 might be more quickly solved as 106 – 30 + 1 if you have the number sense to subtract 106 – 30 in one increment. Efficiency is about making judgments. To make judgments, one needs to know with understanding more than one strategy to solve a problem. Since each strategy is based on slightly different mathematical ideas, knowing more than one strategy means you have a broader base of mathematical understanding.

Learning the Strategies in a Pluralistic Strategy Environment

The various strategies listed below offer a range of options for students to consider. There are noted issues of students and teachers with those curriculums where one strategy is introduced on Monday, a different one on Tuesday, and yet another on Wednesday, and by Friday everyone is confused and frustrated. The goal of this approach is to have students get *anchored in one strategy first before attempting another strategy. However, in public sharing, students are hearing their peers describe how other strategies work.* A sequence of how this approach is enacted follows.

- Introduce a problem for students to solve
- Let students solve it in a manner that makes sense to them
- In public sharing, draw out, name, and compare and contrast the various strategies
- Introduce another problem for students to solve
    - Direct students to pick which strategy makes most sense to them *and use that one even if it is different than what your neighbor chooses*
- In public sharing, focus conversation on how the different strategies work and why it makes sense to its various users.
    - Over time, have students who don't use a particular strategy ask questions of those who use it to help elaborate and clarify how the strategy works.
- After a period of confidence building in being anchored in one strategy and listening to others…
    - Introduce a new problem to solve
    - Direct students to use their favorite strategy first
    - Once they have a solution, do the same problem using a second strategy.

- ○ If stuck or have questions about using a particular strategy, seek out someone in the classroom who uses the strategy and have them help

The important consideration is to have students become knowledgeable of and fluent with one strategy first. Then over a period of weeks, students can take on other strategies that they have seen other peers use and explain. There is no need to rush students to take on more than one strategy.

Multi-digit Addition & Subtraction Strategies

Incremental or Counting Up Strategy [Addition]

Version 1a & 1b

a)	b)
86	86
+ 38	+ 38
116	110
+ 4	+ 8
120	118
+ 4	+ 2
124	120
	+ 4
	124

Script:

- *Version a:* 86 + 30 = 116.
  - 8 is then decomposed into 4 + 4 in order to reassociate one of the fours to the 116 in order to get to the next landmark of ten [120] and then adding the last segment, 4, to that to get the answer of 124.
- *Version b:* 80 + 30 = 110
  - The 8 is incremented on to make 118
  - The 6 is decomposed into 2 + 4 in order to add the 2 to 118 and get to the next landmark of ten [120]. The last increment, 4, is then added to 120 to get the total of 124.

What's in common? A child who works this way is *looking at the numbers as a series of increments that, when combined, creates a running total*. Decomposition options include decomposing either both or just one of them [86 and 38] by their place value components as well as decomposing single-digit numbers into a combination that will allow the reconfiguration of a new ten [116 + 4 or 118 + 2]

Incremental or Counting Back Strategy [Subtraction]

Version 1

86
- 38
56
- 6
50
- 2
48

Script:

- *86 minus 30 is 56.*
- *8 = 6+2 so I will subtract it in two parts,* 56 – 6 *is 50 (that's 6 of the 8), minus the 2 (the last of the 8) is 48. The answer is 48.*

Here the child keeps the larger number whole and takes off the tens and then the ones of the bottom number. This saves time compared to the first versions, but the mathematical ideas are the same.

What's in common? As in addition, the child is removing increments of the amount needed to be separated from the original total. [Subtraction viewed as "take away"] Decompositions are based on place value [38 = 30 + 8] as well as single-digit decompositions in order to *get back to a ten, a decade.*

Early Versions: When children first try these strategies, they may need to count back one ten at a time (*86, 76, 66, 56, (pause) 55, 54, 53…48).* This is normal! As their number sense matures, and they are pushed to think about saving themselves time, they come to work in bigger and bigger chunks of numbers and become very fast

Version 2a & 2b

```
a) 86     b) 86
 - 38       - 38
   50         50
 -  8       +  6
   42         56
 +  6       -  6
   48         50
            -  2
              48
```

Script:

- *80 minus 30 is 50.*
- Now you have two choices, deal with the 8 first or deal with the 6 first.
    - *a)* The 8 first: *50 – 8 is 42. I have to add the six back on so 42 + 6 is 48. The answer is 48.*
    - *b)* Now the 6 first: *50+6 is 56; 8 = 6+2 so* 56 – 6 *is 50, minus 2 is 48. The answer is 48.*

What's in common? A child who works this way is looking at 86 and 38 by their place value components. The child also is exploring the idea that the order of the operations can be altered. There is no limitation here in that regard. But instead of breaking 86 into 70 + 16 as in the standard algorithm, the child is breaking the 38 into 30 + 6 + 2 (version 1b). The adage of, "If you don't like the numbers you have, change them!" is being used here just as in the Standard Algorithm, it's just that the numbers to be altered is the bottom number not the top number. Both choices work!

Combining Like Units, Tens & Ones Strategy, or Partial Sums [Addition]

```
   86
 + 38
  110
 + 14
  124
```

Script:

- *80 plus 30 is 110.*
- *6 plus 8 is 14*
- *110 plus 14 is 124. The answer is 124.*

The decomposition of both numbers is by place value [86 = 80 + 6; 38 = 30 + 8]. The tens are combined. The ones are combined. Given that there is a new ten, the tens are combined, etc. to determine the total answer. What distinguishes Combining Like Units from Incremental is that in this strategy the various units (the tens, the ones, etc.) are combined in like combinations first *before* being combined to find a total sum. In the Incremental Strategy, a running total of subparts are combined to find the total.

The Standard Algorithm - Combining Like Units is the basis for the Standard Algorithm in Addition. The standard algorithm combines the ones first, then the tens in order to find the total sum. The commutative property of addition allows the same answer to be determined regardless of which place is combined first. Whether or not digits are floated on top (American custom) or on the line (European custom), or if the partial sums are shown down below *is irrelevant to the mathematics. Those are issues of formatting only, not the structure of the mathematics.*

Combining Like Units, Tens & Ones Strategy, or Partial Differences [Subtraction]

Scenario 1: *"Going Negative"*

```
  86
- 38
  50
-  2
  48
```

Script:
- *80 minus 30 is 50.*
- *6 minus 8 is negative 2 ($^-2$)*
- *50 and $^-2$ is 48. The answer is 48.*

Note, technically the student is adding 50 + $^-$*2,* but that has the same effect as 50 – 2. This involves knowing about *inverse elements.* It is recommended that one lets that sit in the background and let students use the equivalent positive expression.

Scenario 2: *"What am I short?"*

```
  86
– 38
  50
–  2
  48
```

Script:
- *80 minus 30 is 50.*
- *6 minus 8, I can take away 6 but I'm short 2, so the 2 has to come out of the 50. 50 minus 2 is 48. The answer is 48.*

So, if the student is not comfortable thinking about negative numbers, the student is usually very aware what he or she is short of if they have six and want eight.

Compensation, or "The Rounding Strategy" [Addition]

Version 1: Rounding the bottom number; adjust at the end

```
  86                    86
+ 38  round off to:   + 40
                       126
                      -  2
                       124
```

Script:
- *I round off 38 to 40 to make the problem easier*
- *86 plus 40 is 126.*
- *But with 40 I added 2 too many so I need to subtract 2 so 126 minus 2 is 124. The answer is 124.*

Notice how this is different than in subtraction (below). By making the number larger, more was added. This new total is now unequal to the original expression. [86 + 38 ≠ 86 + 40]. In order to bring the equation back into equilibrium the original amount added must now be taken away [86 + 38 = 86 + (40 - 2)]

Version 2: Translation (Reassociate in advance)

```
  86   subtract 2 from 86    84
+ 38   add 2 to 38         + 40
                            124
```

Script:
- *In addition, I can subtract 2 from the 86 and add that amount to the 38 and still have the same amount as the original.* 86 + 38 = 84 + 40. The new combination, 84 + 40, is easier to work with.

Notice that this compensation strategy is solely based on the associative property of addition. Equality in the equation is maintained at all times [86 + 38 = 84 + 40] unlike version one where the initial adjustment makes the equation unequal requiring an adjustment at the end.

Compensation, or "The Rounding Strategy" [Subtraction]

Version 1: Rounding the bottom number

```
  86                     86
– 38  round off to:    – 40
                         46
                       +  2
                         48
```

Script:

- *I round off 38 to 40 to make the problem easier*
- *86 minus 40 is 46.*
- *But with 40 I took 2 too many so I need to add 2 back on so 46 plus 2 is 48. The answer is 48.*

Notice how this is different than in addition. By making the number larger, more was taken away. If you look at this on a number line, by adding 2 you *shorten the difference between* the two numbers, so you need to add the two back on to lengthen the distance to the original.

Version 2: Maintaining the Difference Between

```
  86   add 2 to 86      88
– 38   add 2 to 38    – 40
                        48
```

Script:

- *If I add 2 to both numbers I get 88 – 40 which will have the same distance between as 86 – 38. 88 minus 40 is 48. The answer is 48.*

Notice that this compensation strategy is solely based upon the *difference between* two numbers rather than the *take away* model with which we are more accustomed. 86 – 38 = 88 – 40. Look at this on a number line to see why this works all the time. Important ideas of equivalency are being developed.

The Difference Between – Adding Up or Subtracting Back To [Subtraction]

Version 1: Add Up to Subtract

```
  86
- 38  Change to 38 + ___ = 86
```

$38 \xrightarrow{+2} 40 \xrightarrow{+46} 86$

46 + 2 = 48

The answer is 48

or

$38 \xrightarrow{+40} 78 \xrightarrow{+2} 80 \xrightarrow{+6} 86$,

The answer is 48

Script:

- I don't like to subtract, so I will turn it into an "adding up to subtract" problem. Add 2 to get to 40, plus 46 gets me to 86. 46 + 2 = 48
- Note that in an earlier version of this strategy, a student will have to add one ten at a time to get to 80, then add the 6. As their number sense grows, students do this in more efficient jumps.
- The second version is premised on students who like to make the big jump first and then the smaller increments

Done vertically

$$\begin{array}{r} 86\underline{\quad}^{+46} \\ -\underline{38}\ \ ^{+2} \\ 48 \end{array}$$

- Vertically: The line drawn between the two numbers is only serving as a visual marker to record the increments. The line represents the "Next Ten" or 40. It saves time in having to rewrite the problem out using the open number line format.
- *This vertical format should only be introduced to students after they have become comfortable and fluent with the number line representation*

Version 2: *Count Back To*

$$\begin{array}{r} 86 \\ -\underline{\ 38} \end{array}$$

$38 \xleftarrow{-2} 40 \xleftarrow{-40} 80 \xleftarrow{-6} 86$

40 + 2 + 6 = 48

The answer is 48

Script:

- This process involves counting back in increments until you get *to* the second number. The equation that matches the structure of this strategy is 86 – ___ = 38.

American Standard Subtraction Algorithm or Trades First Strategy [Subtraction]

Version 1: *Classic Single Digit Language Version*

$$\begin{array}{r} {}^{7}\ \ \\ 8^{1}6 \\ -\underline{\ 3\ 8} \\ 4\ 8 \end{array}$$

Classic Verbal Script:

- *I can't take 8 from 6 so I borrow 1 from the 8, make it a 7, put the 1 in front of the 6 .*
- *16 take away 8 is 8. Put down the 8. 7 take away 3 is 4. Put down the 4. The answer is 48*

Version 2: *Using the Language of Values*

$$\begin{array}{r} 86 = 70 + 16 \\ -\underline{38} = \underline{30 + \ \ 8} \\ 48 = 40 + \ \ 8 \end{array}$$

New Script:

- I don't have enough to take 8 from 6 so *I am going to break apart 86 into 70 and 16 to make 86 easier to work with. I also break 38 into 30 and 8.*
- *16 minus 8 is 8. 70 minus 30 is 40. 40 plus 8 is 48. The answer is 48.*

This is the standard algorithm with understanding. This can be collapsed into looking exactly like the standard algorithm to save space but the script remains a description of values rather than digits.

Other countries in the world have other "Standard Algorithms" that are different from ours. The goal is to think flexibly and efficiently.

The Argument for Knowing Multiple Strategies

There is a benefit to knowing more than one strategy to add and subtract both single-and multi-digit numbers. Each strategy relies on mathematical concepts and skills that are different from the others. Having dexterity using more than one strategy, therefore, broadens the range of mathematical knowledge one can draw upon. Being flexible and agile in one's thinking is a benefit in problem solving. Being adept with multiple solution strategies fosters that type of flexible thinking.

Efficiency is based on making judgments given the numbers in the problem and context governing the situation with the least amount of effort. If I am adept in using multiple strategies, and the numbers presented are 410 - 99, it is more efficient to either solve by using the Compensation Strategy [410 - 100 + 1] of the Adding Up To Subtract Strategy [+1 → 100 + 310 → 410; The answer is 311] then using the American standard algorithm for subtraction with its "borrowing and carrying" across two places. While one can become quite accomplished and speedy using the standard algorithm, the other two strategies involve less cognitive workload. They are more efficient.

Note that among the strategies noted above, there are far more subtraction strategy options than addition. Given the nature of the operation, there are more opportunities for decomposition and reconfiguring numbers, more flexible options given the order of operations, and more options to restructure the problem [Adding up to Subtract or Subtracting Back To] for 'result unknown' contexts.